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Star products on extended massive non-rotating BTZ black holes

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Abstract: AdS_3 space-time admits a foliation by two-dimensional twisted conjugacy classes, stable under the identification subgroup yielding the non-rotating massive BTZ black hole. Each leaf constitutes a classical solution of the space-time Dirac-Born-Infeld action, describing an open D-string in AdS_3 or a D-string winding around the black hole. We first describe two nonequivalent maximal extensions of the non-rotating massive BTZ space-time and observe that in one of them, each D-string worldsheet admits an action of a two-parameter subgroup ($\mathcal{AN}$) of $\mathrm{SL}(2,\mathbb{R})$. We then construct non-formal, $\mathcal{AN}$-invariant, star products that deform the classical algebra of functions on the D-string worldsheets and on their embedding space-times. We end by giving the first elements towards the definition of a Connes spectral triple on non-commutative AdS space-times.

Keywords: BTZ black hole, D-strings, star products, spectral triples .

Dedicated to...
Marguerite.

*"Chercheur FRIA", Belgium
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Contents

1. Introduction

Since its discovery [1], the BTZ black hole solution of 2+1 dimensional gravity has proven to be a useful tool for exploring fundamental issues in black hole physics, both at the classical and quantum levels. Perhaps one of the most remarkable properties of the BTZ solution comes from its connection with string theory. On the one hand, the BTZ black hole can be represented as a quotient of the group manifold $\widetilde{\mathrm{SL}(2,\mathbb{R})} \simeq AdS_3$ by a discrete subgroup of its isometry group $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \widetilde{\mathrm{SL}(2,\mathbb{R})}$ [2, 3]. On the other hand, a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ WZW model is an exact string theory background, describing the propagation of strings on the group manifold. Hence, quotienting out the appropriate discrete subgroup in this model leads to a theory that corresponds to an *exact* string theoretical representation of the BTZ black hole [4, 5].

The BTZ space-time reveals a rich geometric structure. In the generic case ($0 \leq J < M$), AdS_3 space-time admits a global foliation by two dimensional leaves stable under the action of the identification subgroups leading to the black hole [6, 7]. Furthermore, the universal covering space of a generic BTZ space-time realized as some open domain in AdS_3 is, in a canonical way, the total space of a principal fibration over $\mathbb{R}$ with as structure group a minimal parabolic subgroup $\mathcal{AN}$ of $\widetilde{SL(2,\mathbb{R})}$. The action of the structure group is isometric with respect to the AdS_3 metric on the total space. Moreover, each of the fibers is canonically endowed with a Poisson structure, constituting the primary input for the construction of star products, i.e. deformations of the pointwise multiplication of functions defined on the leaves.

For spinless BTZ black holes, the stable leaves are twisted conjugacy classes in $\widetilde{SL(2,\mathbb{R})}$ [6]. Such classes are known to be WZW branes in AdS_3 [8, 9, 10], more precisely, they are extremal for the Dirac-Born-Infeld (DBI) brane action associated to a specific 2-form B on AdS_3 (referred hereafter as the 'B-field'). Consequently, they may be interpreted as *closed D1-branes (D-strings)* winding around the black hole.

For this reason, we restrict ourselves in the present work to the non-rotating massive BTZ black hole. We define deformations of the algebra of functions on BTZ spaces, supported on (or tangential to) the leaves, and require these deformations to be non-formal ('strict' in the sense of Rieffel, see below) and compatible with the action of the structure group $\mathcal{AN}$. Our motivation is threefold. First, a deformation of the brane in the direction of the B-field is generally understood as the (non-commutative) geometrical framework for studying interactions of strings with endpoints attached to the brane [11, 12, 13]. Though curved non-commutative situations have been extensively studied in the context of strict deformation theory *i.e.* in a purely operator algebraic framework (for a review and references, see [14]), non-commutative spaces emerging from string theory have up to now mainly been studied in the case of constant B-fields in flat (Minkowski) backgrounds.

The second motivation relies on the work of Connes and Lott [15], who used Rieffel's strict deformation method for actions of tori to define spectral triples for non-commutative spherical manifolds. The main point of their construction is that the data of an *isometric* action of a torus on a spin manifold yields not only a strict deformation of its function algebra but a compatible deformation of the Dirac operator as well. Therefore, obtaining a strict deformation formula for actions of $\mathcal{AN}$, in the context of BTZ spaces, would yield an example of spectral triple for non-commutative non-compact Lorentzian manifolds with constant curvature, with the additional feature that the deformation would be supported on the $\mathcal{AN}$-orbits. The difficulty here is of course that these orbits cannot be obtained as orbits of an isometric action of $\mathbb{R}^d$.

Finally, as the maximal fibration preserving isometry group (the universal cover of) of a BTZ space is the minimal parabolic group $\mathcal{AN}$, it is natural to ask for a deformation which is invariant under the action of $\mathcal{AN}$. But there is also a deeper reason for requiring this invariance. On the basis of considerations on black hole entropy or just by geometrical interest, one could be tempted to define higher genus locally AdS_3 black holes (every BTZ black hole is topologically $S^1 \times \mathbb{R}^2$). This of course implies implementing the action of a Fuschian group in the BTZ picture. At the classical level, this possibility has already been investigated [16]. Our point here is that it may also be investigated at the deformed level. Indeed, the symmetry group $\mathcal{AN}$ is certainly too small to contain large Fuschian groups, but if the deformation is already $\mathcal{AN}$-invariant, the (classical) action of $\mathcal{AN}$ could perhaps be extended to a (deformed) action of the entire AdS_3 by automorphisms of the deformed algebra. If this is the case (and it is), an action of every Fuschian group is obtained at the deformed functional level. These issues will however not be tackled in the present work.

This paper is organized as follows. In section 2, we recall some geometrical properties of the non-rotating massive BTZ black hole, and describe two nonequivalent maximal extensions of the BTZ space-time. We focus on one of the extensions, where each leaf of

the foliation admits an action of $\mathcal{AN}$. In section 3, we study the space-time properties of the winding D-strings in the extended BTZ space using the DBI action. Section 4 is devoted to star products. We first discuss general properties of invariant star products on group manifolds and how they induce star products on manifolds admitting an action of this group. We then construct a family of star products on the group $\mathcal{AN}$ and on the winding D1-branes' worldvolumes in AdS_3 and BTZ space-times. In section 5, deformed Dirac operators are defined in view of obtaining Connes' spectral triples. Section 6 contains conclusions and perspectives.

2. Geometry of the extended BTZ black holes

We have previously shown [6] that BTZ black holes are canonically endowed with a regular Poisson structure, admitting a characteristic foliation constituted by the orbits of an external bi-action. We shall briefly summarize this construction. The Lorentzian space AdS_3 is defined as the universal covering of the group:

$$\mathrm{SL}(2,\mathbb{R}) = \{\, \mathbf{z} = \begin{pmatrix} u+x & y+t \\ y-t & u-x \end{pmatrix} | x, y, u, t \in \mathbb{R},\ \det \mathbf{z} = 1\} \qquad . \tag{2.1}$$

Its Lie algebra

$$\mathfrak{sl}(2,\mathbb{R}) = \{ \begin{pmatrix} z^H & z^E \\ z^F & -z^H \end{pmatrix} := z^H\,\mathbf{H} + z^E\,\mathbf{E} + z^F\,\mathbf{F}\} \qquad , \tag{2.2}$$

is expressed in terms of the generators $\{\,\mathbf{H},\,\mathbf{E},\,\mathbf{F}\}$ satisfying the commutation relations:

$$[\,\mathbf{H},\,\mathbf{E}] = 2\,\mathbf{E} \quad , \quad [\,\mathbf{H},\,\mathbf{F}] = -2\,\mathbf{F} \quad , \quad [\,\mathbf{E},\,\mathbf{F}] = \,\mathbf{H} \quad . \tag{2.3}$$

By identifying $\mathfrak{sl}(2,\mathbb{R})$ with the tangent space of $\mathrm{SL}(2,\mathbb{R})$ at the identity element $\mathbf{e}$, the Killing metric at this point, denoted by $\beta_{\mathbf{e}}$, is given by:

$$\beta_{\mathbf{e}}(\,\mathbf{X},\,\mathbf{Y}) := \frac{1}{2}\mathrm{Tr}\,(\,\mathbf{X}\,\mathbf{Y}) \qquad ; \qquad \mathbf{X}\,,\,\mathbf{Y} \in \mathfrak{sl}(2,\mathbb{R})\ . \tag{2.4}$$

The Lorentzian character of AdS_3 means that there is an isomorphism between $\mathfrak{sl}(2,\mathbb{R})$ and $M^{2,\,1}$, the Minkowski space in 2+1 dimensions. The generator $\mathbf{H}$ is space-like whereas $\mathbf{E}$ and $\mathbf{F}$ are light-like, since:

$$\beta_{\mathbf{e}}(\,\mathbf{H}\,,\,\mathbf{H}) = 1 \qquad , \qquad \beta_{\mathbf{e}}(\,\mathbf{E}\,,\,\mathbf{F}) = 1/2 \qquad , \qquad \beta_{\mathbf{e}}(\,\mathbf{E}\,,\,\mathbf{E}) = 0 = \beta_{\mathbf{e}}(\,\mathbf{F}\,,\,\mathbf{F}) \qquad . \tag{2.5}$$

Let us also introduce the unit generator $\mathbf{T} = \mathbf{E} - \mathbf{F}$, and the $\mathrm{SL}(2,\mathbb{R})$ subgroups:

$$\mathcal{A} = \exp(\mathbb{R}\ \mathbf{H})\ , \quad \mathcal{N} = \exp(\mathbb{R}\ \mathbf{E})\ , \quad \mathcal{K} = \exp(\mathbb{I}\ \mathbf{T})\ , \qquad \mathbb{I} = [0\,,2\,\pi] \qquad , \tag{2.6}$$

which are the building blocks of the Iwazawa decomposition of $\mathrm{SL}(2,\mathbb{R}) = \mathcal{K}\,\mathcal{A}\,\mathcal{N}$.

The automorphism group of $\mathfrak{sl}(2,\mathbb{R})$ is isomorphic to the three-dimensional Lorentz group $SO(2,\,1) \equiv L_+^{\uparrow}(2,\,1) \cup L_-^{\downarrow}(2,\,1)$. Transformations belonging to $L_+^{\uparrow}(2,\,1)$ correspond

to internal isomorphisms; those belonging to $L^{\downarrow}_{-}(2,\,1)$ need the introduction of an external automorphism that we choose as:

$$\sigma(\,\mathbf{H}) = \mathbf{H} \qquad , \qquad \sigma(\,\mathbf{E}) = -\,\mathbf{E} \qquad , \qquad \sigma(\,\mathbf{F}) = -\,\mathbf{F} \qquad , \tag{2.7}$$

or equivalently, using the matrix representation (2.2):

$$\sigma(\,\mathbf{Z}) = \mathbf{H}\;\mathbf{Z}\;\mathbf{H} \qquad , \qquad \mathbf{Z} \in \mathfrak{sl}(2,\mathbb{R}) \qquad . \tag{2.8}$$

A massive non-rotating BTZ black hole is obtained as the equivalence classes of the quotient of $\mathrm{SL}(2,\mathbb{R})$[1] under the bi-action of the subgroup $\mathcal{H}$, defined by

$$\mathcal{H}\times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R}) : (\mathbf{h},\;\mathbf{z}) \mapsto \mathbf{h}\,\mathbf{z}\,\sigma(\mathbf{h}^{-1}) := \Sigma^{\sigma}_{\mathbf{h}}(\mathbf{z}) \tag{2.9}$$

where the subgroup $\mathcal{H}$ is the restriction of $\mathcal{A}$ to integer values of its parameter:

$$\mathcal{H} = \exp(\pi\sqrt{M}\,\mathbb{Z}\,\mathbf{H}) \qquad , \tag{2.10}$$

where M is the mass of the black hole.

The $\mathrm{SL}(2,\mathbb{R})$ group can also be viewed as the quadric $Q \equiv t^2+u^2-x^2-y^2=1$ in the four-dimensional flat ultra-hyperbolic space $M^{2,\,2}$, with metric $ds^2=-dt^2-du^2+dx^2+dy^2$, while the orbit of a point z under the $\mathrm{SL}(2,\mathbb{R})$ bi-action:

$$\mathcal{O}_z = \{\mathbf{g}\,z\,\sigma(\mathbf{g}^{-1})|\,\mathbf{g}\in \mathrm{SL}(2,\mathbb{R})\} \tag{2.11}$$

is given by the intersection of Q with the planes of constant x coordinate. Accordingly, these orbits are two-dimensional hyperboloids, isomorphic to the unit hyperboloid $\mathbb{H} = \{\,\mathbf{X}\in\mathfrak{sl}(2,\mathbb{R})|\beta_{\mathbf{e}}(\,\mathbf{X},\;\mathbf{X})=1\}$ of $\mathfrak{sl}(2,\mathbb{R})$. The group $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ by the adjoint action. This allows to associate to each point $\mathbf{X}$ of $\mathbb{H}$, expressed as $\mathbf{X}=\mathbf{g}_0\,\mathbf{H}\,\mathbf{g}_0^{-1}$, the equivalence class of matrices:

$$[\mathbf{g}_0] = [\;\exp(\mathbb{R}\;\mathbf{X})\,\mathbf{g}_0\;\exp(\mathbb{R}\;\mathbf{H})\;] = [\mathbf{g}_0\;\exp(\mathbb{R}\;\mathbf{H})\;] \qquad . \tag{2.12}$$

Now it is easy to check that we may represent Q, *i.e.* $\mathrm{SL}(2,\mathbb{R})$, as the product $\mathbb{H}\times\mathbb{R}$ by considering the union of the orbits $\mathcal{O}_{\exp(\mathbb{R}\;\mathbf{H})}$. Indeed, elementary calculations show that $\mathbf{g}_1\exp(\rho_1\;\mathbf{H})\sigma(\mathbf{g}_1^{-1}) = \mathbf{g}_2\exp(\rho_2\;\mathbf{H})\sigma(\mathbf{g}_2^{-1})$ implies that $\rho_1=\rho_2$ and $[\mathbf{g}_1]=[\mathbf{g}_2]$.

The transformations defined by eqs (2.10),(2.9) constitute a discrete isometry subgroup of $\mathrm{SL}(2,\mathbb{R})$, first introduced in ref. [3], and that hereafter we call the BHTZ subgroup. Let us emphasize that the orbits 2.11 are stable under the action of the BHTZ subgroup. This remark, with the isomorphism $\mathrm{SL}(2,\mathbb{R})\simeq\mathbb{H}\times\mathbb{R}$, are the key ingredients to obtain a global coordinate system on BTZ black holes. The generator of the transformation $\Sigma^{\sigma}_{\exp(\mathbb{R}\;\mathbf{H})}(z)$ partitions AdS_3 into different connected components according to its nature. On two connected regions, the generator is time-like. We denote these domains $\mathbf{I}$ and $\mathbf{I}'$. Moreover, there are four connected regions where the generator is space-like. They

[1] We may restrict ourselves to $\mathrm{SL}(2,\mathbb{R})$ as we focus on non-rotating black holes; for rotating black holes, $AdS_3=\widetilde{\mathrm{SL}(2,\mathbb{R})}$ has to be considered; see ref. [7] for details.

are denoted by $\mathbf{II_R}$, $\mathbf{II_L}$ and $\mathbf{III_R}$, $\mathbf{III_L}$ (see Fig.1). On the boundaries between these domains, the generator is light-like or zero. The causally safe domains $\mathbf{I}$ and $\mathbf{I}'$ correspond to $t(z)^2 - y(z)^2 > 0$ [6]. Expressing z as $\Sigma^\sigma_{\mathbf{g}}(\exp(\rho\,\mathbf{H}))$ for suitably chosen $\mathbf{g}$ and ρ, we may express this condition in terms of $\mathbf{g}$ as:

$$1 > t(\mathbf{g})^2 - y(\mathbf{g})^2 > 0 \qquad . \tag{2.13}$$

Let us notice that this last condition is ρ-independent, as expected. Writing $\mathbf{X} = \mathbf{g}\;\mathbf{H}\,\mathbf{g}^{-1} = x^H\,\mathbf{H} + x^E\,\mathbf{E} + x^F\,\mathbf{F}$, this condition implies that:

$$|x^H| < 1 \qquad . \tag{2.14}$$

In ref. [6] we used the following parameterization :

$$\mathbf{X} = \exp(\frac{\theta}{2}\,\mathbf{H})\exp(-\frac{\tau}{2}\,\mathbf{T})\;\mathbf{H}\;\exp(\frac{\tau}{2}\,\mathbf{T})\exp(-\frac{\theta}{2}\,\mathbf{H}) \tag{2.15}$$

leading to the expression of the metric on $\mathbf{I}$ (or $\mathbf{I}'$)

$$ds^2 = L^2\left(d\rho^2 + \cosh(\rho)^2(-d\tau^2 + \sin^2\tau\,d\theta^2)\right) \qquad , \tag{2.16}$$

where we introduced the length scale parameter L related to the cosmological constant Λ by $\Lambda = -L^{-2}$. The boundary of a connected component of this domain is given by the surfaces $\tau = 0,\ \tau = \pi$ or $\tau = \pi,\ \tau = 2\pi$, which correspond to coordinate singularities due to the occurrence of closed light-like curves. Hereafter we restrict ourselves to the domain $\mathbf{I}$ where τ varies from 0 to π.

The singularities we encounter here are of the type of those described by Misner [17]. On BTZ space, indeed, we also have two families of null geodesics that spiral as they approach the chronological horizons, located at $\tau = 0$ and $\tau = \pi$, also called the BTZ black hole singularities [3]. As explained in [18], we may extend BTZ space across the singularities by selecting one of the families of null geodesics near each component of the singularities. According to the family of null geodesics that is untwisted, it is possible to extend a causally safe domain in four different ways; these extensions are two by two isomorphic. AdS_3 space-time can be viewed as a stack (i.e. a trivial fiber bundle) of leaves ρ=cst. On Fig.1, we have depicted the Penrose diagram of such a leaf and its intersection with the different regions $\mathbf{I}$, $\mathbf{I}'$, $\mathbf{II_R}$, $\mathbf{II_L}$, $\mathbf{III_R}$, $\mathbf{III_L}$. We have also represented the intersection of a leaf with a fundamental domain with respect to the action of the BHTZ subgroup. The two nonequivalent extensions of the BTZ black hole consist of the continuation of the central region ($\mathbf{I}$) by the two right regions $\mathbf{II_R}$ and $\mathbf{III_R}$, (or equivalently the two left ones $\mathbf{II_L}$ and $\mathbf{III_L}$) or by extending the central region via a left and a right region. Despite the pathologies in its causal structure, the former type of extension is particularly interesting because it allows to define the action of an $\mathcal{A}\,\mathcal{N}$ group on each leaf of the foliation. Let us consider, to fix the idea, the domain obtained by the union

$$\mathbf{U} = \mathbf{I} \cup \mathbf{II}_R \cup \mathbf{III}_R \tag{2.17}$$

We may parametrize $\mathbf{X}$ as:

$$\mathbf{X} = \exp(\frac{\phi}{2}\,\mathbf{H})\exp(w\;\mathbf{E})\;\mathbf{S}\;\exp(-w\;\mathbf{E})\exp(-\frac{\phi}{2}\,\mathbf{H}) \tag{2.18}$$

where:

$$\mathbf{S} = \exp(-\frac{\pi}{4}\,\mathbf{T})\;\mathbf{H}\;\exp(\frac{\pi}{4}\,\mathbf{T}) \tag{2.19}$$

which leads to the maximally extended metric on the domain $\mathbf{U}$:

$$ds^2 = L^2\left(d\rho^2 + \cosh(\rho)^2(d\phi^2 - (w\,d\phi + dw)^2)\right) \qquad . \tag{2.20}$$

Restricting ϕ to $[0,\,2\pi\sqrt{M}]$ in (2.18) and (2.20), we obtain a maximally extended spinless BTZ black hole, denoted hereafter by $\widetilde{\mathrm{BTZ}}$. The chronological horizons (which are the BTZ "singularities" and correspond to Cauchy horizons in AdS_3) are located at $w = \pm 1$; the black hole horizons are given by

$$\tanh\rho/2 = \pm\left(\frac{1-\sqrt{1-w^2}}{w}\right) \qquad . \tag{2.21}$$

On the intersection of the domains they cover, the coordinate systems (2.16) and (2.20) are related by

$$w = \cos\tau \quad \text{and} \quad e^{2\phi} = \frac{e^{2\theta}}{\sin\tau}\,. \tag{2.22}$$

3. Symmetric D-branes in $\widetilde{\mathrm{BTZ}}$ space-time

Strings moving in group manifolds are described by the WZW model [19]. The exact conformal invariance of these models is based on the current algebra symmetry, generated by the Lie algebra valued chiral currents $J(z)$ and $\bar{J}(\bar{z})$. A well-understood class of D-brane configurations in WZW models is obtained as solutions of the familiar gluing conditions on the chiral currents

$$J(z) = R\bar{J}(\bar{z}) \tag{3.1}$$

at the boundary of the string worldsheet, where R is a metric preserving Lie algebra automorphism [9, 20]. These gluing conditions describe symmetric D-branes, that is, configurations which preserve conformal invariance and the infinite-dimensional symmetry of the current algebra of the bulk theory . The geometry of the associated branes is encoded in these gluing conditions. Their worldvolumes are shown to lie on (twisted) conjugacy classes in the group manifold and constitute classical solutions of the space-time DBI action [8].

Symmetric D-branes of the $Sl(2,\mathbb{R})$ WZW model are of three types : two-dimensional hyperbolic planes (H_2), de Sitter branes (dS_2) and anti-de Sitter branes (AdS_2). In [10], it was shown that the AdS_2 worldvolumes, corresponding to twisted conjugacy classes, are the only physically relevant classical configurations, solutions of the DBI equations. Indeed, the worldvolume electric field on the dS_2 branes is supercritical, while H_2 branes have euclidean signature and must therefore be interpreted as instantons.

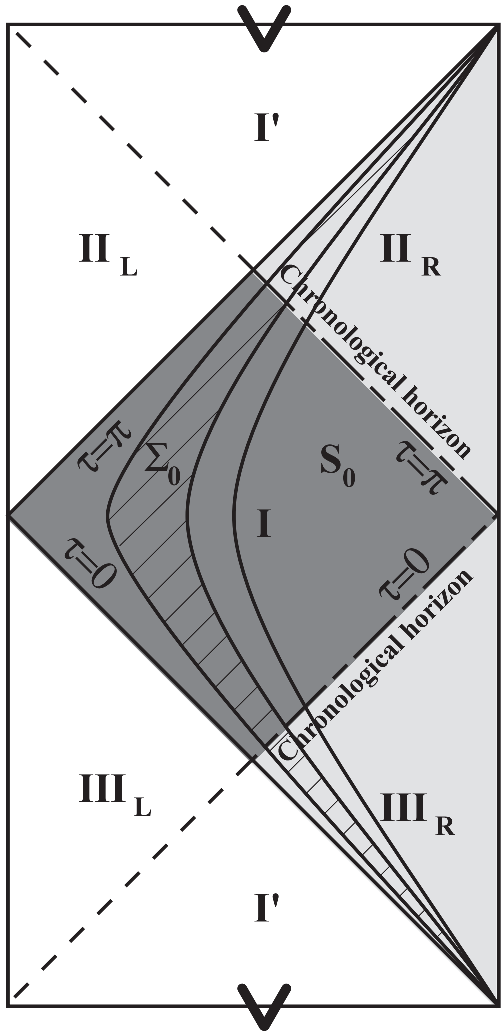

Figure 1: This figure, with the top and bottom lines identified, is a Penrose diagram of a $\rho = \rho_0$ section in $\mathrm{SL}(2,\mathbb{R})$. AdS_3 can be seen as a stack of such fibers (without this identification). Accordingly, this diagram can also be seen as a Penrose diagram of $\mathrm{SL}(2,\mathbb{R})$, each point representing a line parametrized by ρ, running from $-\infty$ to $+\infty$. The region **I** (darkly shaded) provides after identification by the BHTZ subgroup (eq. (2.10)) the usual non-rotating BTZ black hole space-time, bounded by the chronological horizons $\tau = 0$ and $\tau = \pi$. The maximally extended region $\mathbf{S}_0$ (shaded) admits an action of $\mathcal{AN}$, and represents the intersection of the domain $\mathbf{U}$ defined in (2.17) with a $\rho = \rho_0$ section. This maximal extension goes beyond the chronological horizons, where the identifications yielding the black hole become light-like, and which usually referred as the BTZ black hole *chronological singularity* (the BTZ black hole singularity is not a curvature singularity, but merely a singularity in the causal structure). The dashed region Σ_0 represents a fundamental domain of the action of the BHTZ identification subgroup on $\mathbf{S}_0$, i.e. a $\rho = \rho_0$ section of the extended $\widetilde{\mathrm{BTZ}}$ black hole.

We now make a link with the geometry of the non-rotating BTZ black-hole. We saw in the previous section that the spinless BTZ black hole admits a foliation by leaves, the $\rho =$

constant surfaces, which are stable under the action of the BHTZ subgroup and constitutes twisted conjugacy classes in $Sl(2,\mathbb{R})$ (AdS_2 spaces). From our previous discussion, each of these leaves constitutes a D1-brane that is, a solution of the equations of motion derived from the DBI action, which constitutes the effective action for Dp-branes[2] :

$$S_{BI} = T_p \int d^{p+1}x \sqrt{-det(\hat{g} + \hat{B} + 2\pi\alpha' F)} = T_p \int d^{p+1}x \, L_{BI} \quad , \tag{3.2}$$

where $\hat{g}$ and $\hat{B}$ are the pull-backs of the WZW backgrounds and F is the worldvolume electric field. From now on, we set $2\pi\alpha' = 1$. With $K_{ij} = g_{ij} + B_{ij} + F_{ij}$ and $K = \det(K_{ij})$, the equations of motion (for Abelian F-field) derived from the DBI action are :

$$\begin{aligned} \partial_k(\sqrt{-K}K^{(kj)}X^\mu_{,j})g_{\mu\lambda} + \sqrt{-K}K^{(kj)}\Gamma_{\mu\nu,\lambda}X^\mu_{,j}X^\nu_{,k} + \sqrt{-K}K^{[kj]}H_{\mu\nu\lambda}X^\mu_{,j}X^\nu_{,k} &= 0 \\ \partial_i(\sqrt{-K}K^{[ij]}) &= 0 \quad , \end{aligned} \tag{3.3}$$

with $K^{(kj)} = \frac{1}{2}(K^{kj} + K^{jk})$ and $K^{[kj]} = \frac{1}{2}(K^{kj} - K^{jk})$, $H_{\mu\nu\lambda} = \frac{1}{2}(B_{\mu\nu,\lambda} + B_{\nu\lambda,\mu} - B_{\mu\lambda,\nu})$ and $\Gamma_{\mu\nu,\lambda} = \frac{1}{2}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu\lambda})$. Furthermore, in presence of space-time Killing vectors Ξ, such that $\mathcal{L}_\Xi B = d\widetilde{\alpha}$ (a condition trivially satisfied here), the following current is conserved on-shell:

$$J^i = \Xi^\mu \frac{\partial L}{\partial X^\mu_{,i}} - \alpha_j \frac{\partial L}{\partial A_{j,i}} \; , \quad \partial_i J^i \approx 0 \, , \tag{3.4}$$

with $\alpha_j = \widetilde{\alpha}_\mu X^\mu_{,j}$.

The WZW three-form H is proportional to the volume form, the constant of proportionality being fixed by the condition of quantum conformal invariance. We obtain [3]

$$H_{\mu\nu\lambda} = 2\sqrt{-g}\,\varepsilon_{\mu\nu\lambda} \quad . \tag{3.5}$$

A globally defined metric on the extended BTZ space-time is given by (2.20). We fix the gauge by choosing in these coordinates the B-field:

$$B = (\rho + \sinh\rho\cosh\rho)dw \wedge d\phi \quad . \tag{3.6}$$

Solutions of the DBI equations(3.2) read as:

$$\begin{aligned} \rho(x_0,x_1) &= \rho_0, \quad w(x_0,x_1) = x_0, \quad \phi(x_0,x_1) = x_1 \\ F_{01}(x_0,x_1) &= -\rho_0 \quad , \end{aligned} \tag{3.7}$$

where x_0 and x_1 are the worldvolume coordinates on the brane.

These solutions correspond to projections of twisted AdS_3 conjugacy classes that wrap around BTZ space, and are thus compatible with the restriction of the ϕ-variable to its

[2]We have not considered Wess-Zumino terms; they vanish when all Ramond-Ramond background fields are set to zero

[3]The beta function equation reads as [21] $R_{\mu\nu} = \frac{1}{4}H_\mu^{\lambda\omega}H_{\nu\lambda\omega}$. In our case, we have $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$, and fixing the space-time orientation, $k = 2$, see ref. [10].

range $[0, 2\pi\sqrt{M}]$. They may be interpreted as closed DBI 1-branes in BTZ space. In general, the AdS_3 branes obtained from (3.7) by action of isometries do not project into closed branes in BTZ, but into infinite branes that extend from $\rho = -\infty$ to $\rho = +\infty$. Only the AdS_3 branes obtained from isometries compatible with the identifications generated by the BHTZ subgroup lead to closed DBI branes in $\widetilde{\text{BTZ}}$ space. These isometries correspond to the left and right action of the subgroup $\exp(\mathbb{R}\,\mathbf{H})$ on $\mathrm{SL}(2,\mathbb{R})$. In terms of the coordinates (2.20), their Killing vectors read as

$$\Xi_- = \partial_\phi \quad \text{and} \quad \Xi_+ = -w\,\partial_\rho - w\tanh\rho\,\partial_\phi + (w^2-1)\tanh\rho\,\partial_w\,. \tag{3.8}$$

The action of the corresponding isometries on the brane (3.7) yields

$$\begin{aligned}
\sinh\rho(x_0) &= +x_0\cosh\rho_0\sinh f_\pm + \sinh\rho_0\cosh f_\pm \\
w^2(x_0) &= 1 + \frac{\cosh^2\rho_0}{\cosh^2\rho(x_0)}(x_0^2-1) \\
e^{2\phi(x_0,x_1)} &= \frac{\cosh^2\rho(x_0)}{\cosh^2\rho_0}\,e^{2(g_\pm+x_1)}\,,
\end{aligned} \tag{3.9}$$

where $f_\pm$ and $g_\pm$ are constants related to the isometries used to perform the transformations, $f_+ = g_- = 2\pi\sqrt{M}$ and $f_- = g_+ = 0$. On the causally-safe region, these solutions are more easily expressed, using the coordinates of (2.16), as

$$\begin{aligned}
\sinh\rho(x_0) &= \cos x_0\cosh\rho_0\sinh f_\pm + \sinh\rho_0\cosh f_\pm \\
\sin\tau(x_0) &= \frac{\cosh\rho_0\sin x_0}{\cosh\rho(x_0)} \\
\theta(x_1) &= x_1 + g_\pm \quad ,
\end{aligned} \tag{3.10}$$

The corresponding expressions for the worldvolume electric field F_{01} can then be deduced from (3.3). The two constants of motion associated to the solutions (3.7) and (3.9) corresponding to the conserved currents J_+ and J_- [see (3.4) and (3.8)] are

$$J_-^0 = 0 \quad \text{and} \quad J_+^0 = -\sinh\rho_0\,. \tag{3.11}$$

The isometries generated by Ξ_- preserve globally the ρ=cst D1-brane worldvolume. The action simply consists in a rotation of the brane on itself. Those generated by Ξ_+ generate boosts of the brane.

4. Star products

In this section we construct star products on the branes just obtained.

4.1 Induced star products

Here we follow the method described in [22, 23]. Let us remind that a star product is a one-parameter (denoted $\lambda\!\!\!^-$) deformation of the algebra of functions on a (symplectic) manifold with pointwise multiplication, defining an associative composition law on a functional

space that admits left and right unit elements, and such that to first order the deformed commutator provides the Poisson bracket.

Consider a Lie group G. If u is a complex valued function on G, *i.e.* $u \in \mathrm{Fun}[G,\mathbb{C}]$, we denote by $L^\star_{\mathbf{g}}[u]$ (resp. $R^\star_{\mathbf{g}}$) the composition of this function with the left (resp. right) translations on the group:

$$\forall\, \mathbf{g},\, \mathbf{h} \in G, \quad L^\star_{\mathbf{h}}[u](\mathbf{g}) = u(\mathbf{h}\,\mathbf{g}) \quad , \quad R^\star_{\mathbf{h}}[u](\mathbf{g}) = u(\mathbf{g}\,\mathbf{h}) \qquad . \tag{4.1}$$

If the group G also acts on a space X, let say by a left action τ:

$$G \times X \xrightarrow{\tau} X \;:\; (\mathbf{g},\, \mathbf{x}) \mapsto \tau_{\mathbf{g}}(\mathbf{x}) \qquad \text{with} \qquad \tau_{\mathbf{g}\,\mathbf{h}} = \tau_{\mathbf{g}} \circ \tau_{\mathbf{h}} \qquad , \tag{4.2}$$

we may use the induced action of τ on $\mathrm{Fun}[X,\mathbb{C}]$ to define an induced star product, $\underset{X}{\star}$, on X, as follows. We denote by $\alpha_{\mathbf{g}}$ the induced action of τ, defined as

$$\alpha_{\mathbf{g}}[u](\mathbf{x}) = u(\tau_{\mathbf{g}^{-1}}(\mathbf{x})) \quad , \forall\, \mathbf{x} \in X, \quad \alpha_{\mathbf{g}\,\mathbf{h}} = \alpha_{\mathbf{g}} \circ \alpha_{\mathbf{h}} \qquad . \tag{4.3}$$

Accordingly, for fixed $\mathbf{x} \in X$, we may define a map $\tilde{\alpha}^{\mathbf{x}}$ from $\mathrm{Fun}[X,\, \mathbb{C}]$ into $\mathrm{Fun}[G,\, \mathbb{C}]$:

$$\mathrm{Fun}[X,\, \mathbb{C}] \overset{\tilde{\alpha}^{\mathbf{x}}}{\to} \mathrm{Fun}[G,\, \mathbb{C}] :\; u \mapsto \tilde{\alpha}^{\mathbf{x}}[u] \tag{4.4}$$

$$\forall\, \mathbf{g} \in G \;:\; \tilde{\alpha}^{\mathbf{x}}[u](\mathbf{g}) = u(\tau_{\mathbf{g}^{-1}}(\mathbf{x})) \quad . \tag{4.5}$$

Let us also assume that on G we have a left invariant star product, denoted by $\overset{L}{\underset{G}{\star}}$, *i.e.* a star product satisfying the relation

$$L^\star_{\mathbf{g}}[u \overset{L}{\underset{G}{\star}} v] = L^\star_{\mathbf{g}}[u] \overset{L}{\underset{G}{\star}} L^\star_{\mathbf{g}}[v] \qquad . \tag{4.6}$$

From the left invariant star product on G, we induce a star product on X, denoted $\underset{X}{\star}$, defined as :

$$\left(u \underset{X}{\star} v \right)(\mathbf{x}) := \left(\tilde{\alpha}^{\mathbf{x}}[u] \overset{L}{\underset{G}{\star}} \tilde{\alpha}^{\mathbf{x}}[v] \right)(\mathbf{e}) \qquad , \tag{4.7}$$

$\mathbf{e}$ denoting the identity element of G. If, instead of choosing to evaluate the star product on G at the identity, we use the point $\mathbf{g}$, we obtain a composition law related to the first one by the action of $\alpha_{\mathbf{g}}$:

$$\alpha_{\mathbf{g}} \left(u \underset{X}{\star} v \right)(\mathbf{x}) = \left(\tilde{\alpha}^{\mathbf{x}}[u] \overset{L}{\underset{G}{\star}} \tilde{\alpha}^{\mathbf{x}}[v] \right)(\mathbf{g}) \qquad , \tag{4.8}$$

but which is associative only for $\mathbf{g} = \mathbf{e}$. The $\underset{X}{\star}$ of two functions exists only when the r.h.s of (4.7) exists (see discussions on functional space in the following sections).

On the other hand, using the inversion map

$$G \times X \xrightarrow{i} X \;:\; \mathbf{g} \mapsto i(\mathbf{g}) := \mathbf{g}^{-1} \qquad , \tag{4.9}$$

and its induced action on $\mathrm{Fun}[G,\, \mathbb{C}]$:

$$\forall \mathbf{g} \in G, \quad i^\star[u](\mathbf{g}) := u(\mathbf{g}^{-1}) \qquad , \tag{4.10}$$

we may construct a right invariant star product on G starting from the left one:

$$u \overset{R}{\underset{G}{\star}} v := i^\star \left[i^\star[u] \overset{L}{\underset{G}{\star}} i^\star[v] \right] \qquad . \tag{4.11}$$

The right invariance of $\overset{R}{\underset{G}{\star}}$ results immediately from the left invariance of $\overset{L}{\underset{G}{\star}}$ and the relations on Fun$[G, \mathbb{C}]$:

$$R^\star_{\mathbf{g}} \left[i^\star[u] \right] = i^\star \left[L^\star_{\mathbf{g}^{-1}}[u] \right] \quad , \quad L^\star_{\mathbf{g}} \left[i^\star[u] \right] = i^\star \left[R^\star_{\mathbf{g}^{-1}}[u] \right] \qquad . \tag{4.12}$$

Indeed we obtain:

$$R^\star_{\mathbf{g}}[u] \overset{R}{\underset{G}{\star}} R^\star_{\mathbf{g}}[v] = i^\star \left[i^\star \left[R^\star_{\mathbf{g}}[u] \right] \overset{L}{\underset{G}{\star}} i^\star \left[R^\star_{\mathbf{g}}[v] \right] \right] = R^\star_{\mathbf{g}} \left[u \overset{R}{\underset{G}{\star}} v \right] \qquad . \tag{4.13}$$

4.2 $\mathcal{AN}$-invariant star products

In section (3), we have shown that the DBI branes $\rho = \rho_0$, in AdS_3 as well as in a maximally extended BTZ black-hole, are orbits of the action of an $\mathcal{AN}$ group. In this subsection we build left invariant star products on the $\mathcal{AN}$ group, privileging a pragmatic approach whose geometric meaning will be discussed in future work. Other invariant star products can be obtained from these products using the procedure described in section (4.1), and even more general star products are given by eqs (4.74, 4.75).

The group manifold variables are denoted by a and n. The infinitesimal left translations are generated by the vector fields ∂_a and $\exp[-a]\partial_n$. The star product is written as:

$$(u * v)(\mathbf{x}) = \int \mathrm{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]\, u(\mathbf{y})\, v(\mathbf{z})\, d\mu_{\mathbf{y}}\, d\mu_{\mathbf{z}} \tag{4.14}$$

where the (left invariant) measure used is simply $d\mu_{\mathbf{x}} = da_{\mathbf{x}}\, dn_{\mathbf{x}}$. To be left invariant, the kernel $\mathrm{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}]$ has to verify the equations:

$$\left(\partial_{a_{\mathbf{x}}} + \partial_{a_{\mathbf{y}}} + \partial_{a_{\mathbf{z}}} \right) \mathrm{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] = 0 \qquad , \tag{4.15}$$

$$\left(\exp[-a_{\mathbf{x}}]\, \partial_{n_{\mathbf{x}}} + \exp[-a_{\mathbf{y}}]\, \partial_{n_{\mathbf{y}}} + \exp[-a_{\mathbf{z}}]\, \partial_{n_{\mathbf{z}}} \right) \mathrm{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] = 0 \qquad . \tag{4.16}$$

Hence it depends on four variables instead of six:

$$\mathrm{K}[\mathbf{x}, \mathbf{y}, \mathbf{z}] = K^L[\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}] \qquad , \tag{4.17}$$

where we have set

$$\alpha_{\mathbf{x}\,\mathbf{y}} := a_{\mathbf{x}} - a_{\mathbf{y}} \quad , \quad \nu_{\mathbf{x}\,\mathbf{y}} := n_{\mathbf{x}} - \exp[-(a_{\mathbf{x}} - a_{\mathbf{y}})]\, n_{\mathbf{y}} \qquad . \tag{4.18}$$

This condition ensures the left invariance of the star product under the $\mathcal{AN}$ group. We now proceed to impose four additional conditions: the two that define a star product, i.e. the associativity and the existence of right and left unit, as well as a condition on the trace and the hermiticity.

First of all, the existence of a unit element, $u \overset{L}{\underset{G}{\star}} 1 = u$ and $1 \overset{L}{\underset{G}{\star}} u = u$, imposes the conditions:

$$\int K^L[\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}]\, d\mu_{\mathbf{z}} = \delta^2[\mathbf{x} - \mathbf{y}] \qquad , \tag{4.19}$$

$$\int K^L[\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}]\, d\mu_{\mathbf{y}} = \delta^2[\mathbf{x} - \mathbf{z}] \qquad . \tag{4.20}$$

To fulfill these conditions, we assume that:

$$K^L(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}) = \frac{1}{(2\,\pi\lambda\!\!\!^{-})^2} B(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})\, \exp\{i\Psi(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}})\} \tag{4.21}$$

where:

$$\Psi(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}) := Y(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})\, \nu_{\mathbf{y}\,\mathbf{x}} + Z(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})\, \nu_{\mathbf{z}\,\mathbf{x}} \quad , \tag{4.22}$$

with $Y(a,\, b)$ and $Z(a,\, b)$ real functions, and $B(a,\, b)$ complex. This special choice of the phase Ψ, linear in the $\nu_{\mathbf{y}\,\mathbf{x}}$ and $\nu_{\mathbf{z}\,\mathbf{x}}$ variables, as well as the independence of the function B in these variables, is dictated by the structure of the Fourier transform of the Dirac delta distribution. Eq. 4.19 now reads :

$$\frac{1}{2\,\pi\lambda\!\!\!^{-2}} \int B(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) \delta[Z(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})] \exp\{iY(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) \nu_{\mathbf{y}\,\mathbf{x}}\}\, da_{\mathbf{z}} = \delta^2[\mathbf{x} - \mathbf{y}] \qquad . \tag{4.23}$$

To reproduce the distribution of the right hand side of eq. (4.23), we have to assume that:

$$\delta[Z(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})] = \delta[\alpha_{\mathbf{x}\,\mathbf{y}}]/\zeta(\alpha_{\mathbf{x}\,\mathbf{z}}) \quad \text{i. e.} \quad Z(a,\, b) = 0 \quad \text{iff} \quad a = 0 \qquad . \tag{4.24}$$

Hence, to verify eq.(4.19), the function $B(a,\, b)$ has to satisfy the condition:

$$\lambda\!\!\!^{-2}\, \zeta(b)\, \partial_b Y(0, b) = B(0, b) \qquad , \tag{4.25}$$

with the sign being fixed by the requirement that $Y(0, b)$ runs from $-\infty$ to $+\infty$ when b goes from $-\infty$ to $+\infty$. An analogous calculation shows that eq. (4.20) implies:

$$\lambda\!\!\!^{-2}\, \eta(a)\, \partial_a Z(a, 0) = B(a, 0) \qquad , \tag{4.26}$$

and leads to a relation similar to eq. (4.24):

$$\delta[Y(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})] = \delta[\alpha_{\mathbf{x}\,\mathbf{z}}]/\eta(\alpha_{\mathbf{x}\,\mathbf{y}}) \quad \text{i. e.} \quad Y(a,\, b) = 0 \quad \text{iff} \quad b = 0 \qquad , \tag{4.27}$$

with $Z(a, 0)$ running from $-\infty$ to $+\infty$ when a varies from $-\infty$ to $+\infty$.

Let us now impose the following trace condition on the star product :

$$\int (u \overset{L}{\underset{G}{\star}} v)(\mathbf{x}) d\mu_{\mathbf{x}} := \int k[\mathbf{y},\, \mathbf{z}] u(\mathbf{y})\, v(\mathbf{z}) d\mu_{\mathbf{y}} d\mu_{\mathbf{z}} = \int k[\mathbf{z},\, \mathbf{y}] u(\mathbf{y})\, v(\mathbf{z}) d\mu_{\mathbf{y}} d\mu_{\mathbf{z}} \qquad , \tag{4.28}$$

which implies that the two-point kernel:

$$\int K^L[\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}; \nu_{\mathbf{y}\,\mathbf{x}}, \nu_{\mathbf{z}\,\mathbf{x}}]\, d\mu_{\mathbf{x}} := k[\mathbf{y},\, \mathbf{z}] = k[\mathbf{z},\, \mathbf{y}] \tag{4.29}$$

is symmetric. We shall not discuss here this condition in all its generality, but first restrict ourselves to the special case:

$$k[\mathbf{y},\, \mathbf{z}] = \delta^2[\mathbf{y},\, \mathbf{z}] \qquad , \tag{4.30}$$

and discuss a slightly more general situation later. After integration over $n_{\mathbf{x}}$ in (4.29), this condition may be rewritten as:

$$\frac{1}{2\,\pi \lambda\!\!\!^{-2}} \int B(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) \delta[Y(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) \exp(\,\alpha_{\mathbf{x}\,\mathbf{y}}) + Z(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) \exp(\,\alpha_{\mathbf{x}\,\mathbf{z}})]$$
$$\exp\left\{iY(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) n_{\mathbf{y}} + Z(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}}) n_{\mathbf{z}})\right\} da_{\mathbf{x}} = \delta(a_{\mathbf{y}} - a_{\mathbf{z}})\, \delta(n_{\mathbf{y}} - n_{\mathbf{z}}) \quad . \tag{4.31}$$

To satisfy this relation, we assume that the delta distribution appearing in the integrand is equivalent to $\delta[\alpha_{\mathbf{y}\,\mathbf{x}} - \alpha_{\mathbf{z}\,\mathbf{x}}]$. In other words, we assume that:

$$Y(a,\, b)\, \mathrm{e}^{\,a} + Z(a,\, b)\, \mathrm{e}^{\,b} = 0 \Leftrightarrow a = b \qquad . \tag{4.32}$$

This condition is sufficient to pursue the construction of the star product; we shall return to it later.

Now we analyze the conditions implied by the associativity. The star product will be associative if and only if $I_R = I_L$, where:

$$I_R = \int K^L(\mathbf{x}, \mathbf{p}, \mathbf{y})\, K^L(\mathbf{y}, \mathbf{q}, \mathbf{r})\, d\mu_{\mathbf{y}} \qquad \text{and} \qquad I_L = \int K^L(\mathbf{x}, \mathbf{y}', \mathbf{r})\, K^L(\mathbf{y}', \mathbf{p}, \mathbf{q})\, d\mu_{\mathbf{y}'} \qquad . \tag{4.33}$$

After integrations on $n_{\mathbf{y}}$ and $n_{\mathbf{y}'}$ we obtain:

$$\begin{aligned} I_R \;=\; & \frac{1}{8\,\pi^3 \lambda\!\!\!^{-4}} \int B(\alpha_{\mathbf{x}\,\mathbf{p}}, \alpha_{\mathbf{x}\,\mathbf{y}})\, B(\alpha_{\mathbf{y}\,\mathbf{q}}, \alpha_{\mathbf{y}\,\mathbf{r}}) \mathrm{e}^{-i\{(Y(\alpha_{\mathbf{x}\,\mathbf{p}},\alpha_{\mathbf{x}\,\mathbf{y}})\,\mathrm{e}^{\,\alpha_{\mathbf{x}\,\mathbf{p}}} + Z(\alpha_{\mathbf{x}\,\mathbf{p}},\alpha_{\mathbf{x}\,\mathbf{y}})\,\mathrm{e}^{\,\alpha_{\mathbf{x}\,\mathbf{y}}})n_{\mathbf{x}}\}} \\ & \mathrm{e}^{i\{Y(\alpha_{\mathbf{x}\,\mathbf{p}},\alpha_{\mathbf{x}\,\mathbf{y}})\,n_{\mathbf{p}} + Y(\alpha_{\mathbf{y}\,\mathbf{q}},\alpha_{\mathbf{y}\,\mathbf{r}})\,n_{\mathbf{q}} + Z(\alpha_{\mathbf{y}\,\mathbf{q}},\alpha_{\mathbf{y}\,\mathbf{r}})\,n_{\mathbf{r}}\}} \\ & \delta[Z(\alpha_{\mathbf{x}\,\mathbf{p}},\, \alpha_{\mathbf{x}\,\mathbf{y}}) - Y(\alpha_{\mathbf{y}\,\mathbf{q}},\, \alpha_{\mathbf{y}\,\mathbf{r}})\mathrm{e}^{\alpha_{\mathbf{y}\,\mathbf{q}}} - Z(\alpha_{\mathbf{y}\,\mathbf{q}},\, \alpha_{\mathbf{y}\,\mathbf{r}})\mathrm{e}^{\alpha_{\mathbf{y}\,\mathbf{r}}}]\; da_{\mathbf{y}} \end{aligned} \tag{4.34}$$

$$\begin{aligned} I_L \;=\; & \frac{1}{8\,\pi^3 \lambda\!\!\!^{-4}} \int B(\alpha_{\mathbf{x}\,\mathbf{y}'}, \alpha_{\mathbf{x}\,\mathbf{r}})\, B(\alpha_{\mathbf{y}'\,\mathbf{p}}, \alpha_{\mathbf{y}'\,\mathbf{q}}) \mathrm{e}^{-i\left\{\left(Y(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{x}\,\mathbf{r}})\,\mathrm{e}^{\,\alpha_{\mathbf{x}\,\mathbf{y}'}} + Z(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{x}\,\mathbf{r}})\,\mathrm{e}^{\,\alpha_{\mathbf{x}\,\mathbf{r}}}\right)n_{\mathbf{x}}\right\}} \\ & \mathrm{e}^{i\left\{Y(\alpha_{\mathbf{y}'\,\mathbf{p}},\alpha_{\mathbf{y}'\,\mathbf{q}})\,n_{\mathbf{p}} + Z(\alpha_{\mathbf{y}'\,\mathbf{p}},\alpha_{\mathbf{y}'\,\mathbf{q}})\,n_{\mathbf{q}} + Z(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{x}\,\mathbf{r}})\,n_{\mathbf{r}}\right\}} \\ & \delta[Y(\alpha_{\mathbf{x}\,\mathbf{y}'},\, \alpha_{\mathbf{x}\,\mathbf{r}}) - Y(\alpha_{\mathbf{y}'\,\mathbf{p}},\, \alpha_{\mathbf{y}'\,\mathbf{q}})\mathrm{e}^{\alpha_{\mathbf{y}'\,\mathbf{p}}} - Z(\alpha_{\mathbf{y}'\,\mathbf{p}},\, \alpha_{\mathbf{y}'\,\mathbf{q}})\mathrm{e}^{\alpha_{\mathbf{y}'\,\mathbf{q}}}]\; da_{\mathbf{y}'} \end{aligned} \tag{4.35}$$

Using eq. (4.32), we see that points such that $a_{\mathbf{q}} = a_{\mathbf{r}}$ and $a_{\mathbf{p}} = a_{\mathbf{x}}$ belong to the support of the delta distribution appearing in I_R, while those such that $a_{\mathbf{p}} = a_{\mathbf{q}}$ and $a_{\mathbf{r}} = a_{\mathbf{x}}$ belong to the support of the delta distribution appearing in I_L. To obtain $I_R = I_L$, the supports of these delta distributions must coincide (possibly after redefinition of the $a_{\mathbf{y}'}$ variable). To go ahead we assume that the support of both delta distributions are located on the subset defined by:

$$a_{\mathbf{p}} - a_{\mathbf{q}} + a_{\mathbf{r}} - a_{\mathbf{x}} = 0 \qquad . \tag{4.36}$$

This condition and the requirement of the equality of the phases of the two integrals (4.34 and 4.35), leads to the set of four equations:

$$
\begin{aligned}
Y(\alpha_{\mathbf{r}\,\mathbf{q}},\alpha_{\mathbf{x}\,\mathbf{y}}) &= Y(\alpha_{\mathbf{y}'\,\mathbf{p}},\alpha_{\mathbf{y}'\,\mathbf{q}}) && (4.37)\\
Y(\alpha_{\mathbf{y}\,\mathbf{q}},\alpha_{\mathbf{y}\,\mathbf{r}}) &= Z(\alpha_{\mathbf{y}'\,\mathbf{p}},\alpha_{\mathbf{y}'\,\mathbf{q}}) && (4.38)\\
Z(\alpha_{\mathbf{y}\,\mathbf{q}},\alpha_{\mathbf{y}\,\mathbf{r}}) &= Z(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{p}\,\mathbf{q}}) && (4.39)\\
Y(\alpha_{\mathbf{r}\,\mathbf{q}},\alpha_{\mathbf{x}\,\mathbf{y}})\mathrm{e}^{\alpha_{\mathbf{r}\,\mathbf{q}}} + Z(\alpha_{\mathbf{r}\,\mathbf{q}},\alpha_{\mathbf{x}\,\mathbf{y}})\mathrm{e}^{\alpha_{\mathbf{x}\,\mathbf{y}}} &= Y(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{p}\,\mathbf{q}})\mathrm{e}^{\alpha_{\mathbf{x}\,\mathbf{y}'}} + Z(\alpha_{\mathbf{x}\,\mathbf{y}'},\alpha_{\mathbf{p}\,\mathbf{q}})\mathrm{e}^{\alpha_{\mathbf{p}\,\mathbf{q}}} && (4.40)
\end{aligned}
$$

To ensure the compatibility between the functional relations (4.37-4.39) and eqs (4.24, 4.27) we must impose the conditions:

$$
\begin{aligned}
a_{\mathbf{y}} = a_{\mathbf{x}} = a_{\mathbf{p}} - a_{\mathbf{q}} + a_{\mathbf{r}} &\Leftrightarrow a_{\mathbf{y}'} = a_{\mathbf{q}} \qquad , && (4.41)\\
a_{\mathbf{y}} = a_{\mathbf{r}} &\Leftrightarrow a_{\mathbf{y}'} = a_{\mathbf{p}} \qquad , && (4.42)\\
a_{\mathbf{y}} = a_{\mathbf{q}} &\Leftrightarrow a_{\mathbf{y}'} = a_{\mathbf{x}} = a_{\mathbf{p}} + a_{\mathbf{r}} - a_{\mathbf{q}} \qquad . && (4.43)
\end{aligned}
$$

The only linear relation between $a_{\mathbf{y}}$ and $a_{\mathbf{y}'}$ satisfying relations (4.41)-(4.43)is

$$
a_{\mathbf{y}'} = -a_{\mathbf{y}} + a_{\mathbf{p}} + a_{\mathbf{r}} \qquad ; \tag{4.44}
$$

the linearity of the relation is imposed by the equality of the integrals I_R and I_L. Furthermore, by inserting the relations (4.36, 4.44) in eqs (4.37, 4.39) we deduce that the functions Y and Z depend on one variable only:

$$
\begin{aligned}
Y(a,\,b) &= \tilde{Y}(b) \quad \text{and} \quad \tilde{Y}(0) = 0 \qquad , && (4.45)\\
Z(a,\,b) &= \tilde{Z}(a) \quad \text{and} \quad \tilde{Z}(0) = 0 \qquad . && (4.46)
\end{aligned}
$$

Moreover eq. (4.38) implies:

$$
\tilde{Y}(a) = \tilde{Z}(-a) \qquad . \tag{4.47}
$$

Condition (4.40) now becomes:

$$
\begin{aligned}
\tilde{Y}(\alpha_{\mathbf{p}\,\mathbf{q}} + \alpha_{\mathbf{r}\,\mathbf{y}})\ \mathrm{e}^{\alpha_{\mathbf{r}\,\mathbf{q}}} + \tilde{Y}(\alpha_{\mathbf{q}\,\mathbf{r}})\ \mathrm{e}^{(\alpha_{\mathbf{p}\,\mathbf{q}}+\alpha_{\mathbf{r}\,\mathbf{y}})} &= \\
\tilde{Y}(\alpha_{\mathbf{p}\,\mathbf{q}})\ \mathrm{e}^{\alpha_{\mathbf{y}\,\mathbf{q}}} + \tilde{Y}(\alpha_{\mathbf{q}\,\mathbf{y}})\ \mathrm{e}^{\alpha_{\mathbf{p}\,\mathbf{q}}} \qquad . & && (4.48)
\end{aligned}
$$

If in this last condition we set $a_{\mathbf{p}} = a_{\mathbf{y}}$ and $a_{\mathbf{q}} = a_{\mathbf{r}}$, we deduce that $\tilde{Y}$ and $\tilde{Z}$ are odd functions satisfying:

$$
\tilde{Y}(a) = -\tilde{Z}(a) = \tilde{Z}(-a) = -\tilde{Y}(-a) \qquad . \tag{4.49}
$$

Finally, considering the condition (4.48) for $a_{\mathbf{q}} = a_{\mathbf{y}}$ and $a_{\mathbf{p}} = a_{\mathbf{r}}$, we obtain:

$$
\tilde{Y}(2\,a) = 2\,\tilde{Y}(a)\ \cosh(\,a) \qquad , \tag{4.50}
$$

whose only continuous solution, with the condition that $\tilde{Y}(a)$ runs from $-\infty$ to $+\infty$ when a starts from $-\infty$, is:

$$
\tilde{Y}(a) = \lambda\!\!\!^{-}{}^{-1}\sinh(\,a) \qquad , \tag{4.51}
$$

where λ is a positive constant. These considerations allow to fix the Ψ-phase of the kernel defining the star product as:

$$\Psi = \lambda^{-1} \left\{ \sinh\left[(a_{\mathbf{y}} - a_{\mathbf{x}})\right] \, n_{\mathbf{z}} + \sinh\left[(a_{\mathbf{z}} - a_{\mathbf{y}})\right] \, n_{\mathbf{x}} + \sinh\left[(a_{\mathbf{x}} - a_{\mathbf{z}})\right] \, n_{\mathbf{y}} \right\} \qquad . \tag{4.52}$$

Indeed, it is straightforward to check that this expression of Ψ satisfies the assumption (4.36) made about the supports of the delta distributions appearing in eqs (4.34 and 4.35).

Having obtained Ψ, we still have to determine the B-function of the kernel introduced in eq. (4.21). From the existence of right and left units we obtain, using eqs (4.24, 4.27):

$$B(0, \alpha_{\mathbf{x}\,\mathbf{z}}) = \cosh(\,\alpha_{\mathbf{x}\,\mathbf{z}}) \qquad \text{and} \qquad B(\alpha_{\mathbf{x}\,\mathbf{y}}, 0) = \cosh(\,\alpha_{\mathbf{x}\,\mathbf{y}}) \qquad , \tag{4.53}$$

while the trace condition (4.31) implies the diagonal relation:

$$B(\alpha_{\mathbf{x}\,\mathbf{y}},\, \alpha_{\mathbf{x}\,\mathbf{y}}) = \cosh(\,\alpha_{\mathbf{x}\,\mathbf{y}}) \qquad . \tag{4.54}$$

In addition, the associativity condition requires that the B-function obeys the quadratic functional equation:

$$\frac{B(\alpha_{\mathbf{r}\,\mathbf{q}},\, \alpha_{\mathbf{r}\,\mathbf{q}} + \alpha_{\mathbf{p}\,\mathbf{y}})\, B(\alpha_{\mathbf{y}\,\mathbf{q}},\, \alpha_{\mathbf{y}\,\mathbf{r}})}{\cosh(\alpha_{\mathbf{q}\,\mathbf{r}})} = \frac{B(\alpha_{\mathbf{y}\,\mathbf{q}},\, \alpha_{\mathbf{p}\,\mathbf{q}})\, B(\alpha_{\mathbf{r}\,\mathbf{y}},\, \alpha_{\mathbf{r}\,\mathbf{q}} + \alpha_{\mathbf{p}\,\mathbf{y}})}{\cosh(\alpha_{\mathbf{q}\,\mathbf{p}})} \tag{4.55}$$

from which we easily deduce (by considering the two special cases $a_{\mathbf{q}} = a_{\mathbf{r}}$ and $a_{\mathbf{p}} = a_{\mathbf{y}}$) that:

$$B(a, b)\, B(b, a) = \cosh(\,a) \cosh(\,b) \cosh(a - b) \qquad . \tag{4.56}$$

The last condition that we impose on the star product is the hermiticity condition:

$$\overline{\left(u \underset{G}{\overset{L}{\star}} v\right)} = \left(\overline{v} \underset{G}{\overset{L}{\star}} \overline{u}\right) \qquad . \tag{4.57}$$

It implies:

$$B(a,\, b) = \overline{B(b,\, a)} \tag{4.58}$$

and the antisymmetry of Ψ with respect to this exchange, a condition that is already satisfied. So, if we assume the hermiticity condition, we obtain:

$$|B(\alpha_{\mathbf{x}\,\mathbf{y}},\, \alpha_{\mathbf{x}\,\mathbf{z}})| = \sqrt{\cosh(\,\alpha_{\mathbf{x}\,\mathbf{y}}) \cosh(\,\alpha_{\mathbf{x}\,\mathbf{z}}) \cosh(\,\alpha_{\mathbf{y}\,\mathbf{z}})} \qquad . \tag{4.59}$$

The phase $\psi(\alpha_{\mathbf{x}\,\mathbf{y}}, \alpha_{\mathbf{x}\,\mathbf{z}})$ of the complex function B (not to be confused with Ψ)is an arbitrary odd function of two variables, vanishing when one of the variable is an integer multiple of π. Indeed, eqs (4.25, 4.26) imply that $B(a, 0)$ and $B(0, a)$ are real.

$$\psi(a, 0) = 0 \qquad . \tag{4.60}$$

It is a matter of trivial calculations to check that the kernel built with the Ψ-phase (4.52) and the function B so defined provides a hermitian star product, satisfying the trace condition (4.28) and admitting a left and right unit. Let us also notice that both the phase

(4.52) and the amplitude (4.59) admit a geometrical interpretation in terms of geodesic triangles built on the three points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ (see refs [22, 24]).

Another left invariant star product under the $\mathcal{AN}$ group was obtained previously by one of us, starting from completely different considerations [22]. Its phase is also given by eq. (4.52) but $B(\alpha_{\mathbf{x}\,\mathbf{y}},\,\alpha_{\mathbf{x}\,\mathbf{z}})$ is real and given by :

$$B(\alpha_{\mathbf{x}\,\mathbf{y}},\,\alpha_{\mathbf{x}\,\mathbf{z}}) = \cosh(a_{\mathbf{y}} - a_{\mathbf{z}}) \qquad . \tag{4.61}$$

This star product does not satisfy the trace condition (4.28) with (4.30) but instead a twisted trace condition involving a K_0 Bessel function:

$$\int (u \overset{L}{\underset{G}{\star}} v)(\mathbf{x}) d\mu_{\mathbf{x}} = \frac{1}{\pi \lambda\!\!\!^{-}} \int K_0 \left[\lambda\!\!\!^{-\,-1}(n_{\mathbf{y}} - n_{\mathbf{z}})\right] \delta(a_{\mathbf{y}} - a_{\mathbf{z}}) u(\mathbf{y})\, v(\mathbf{z}) d\mu_{\mathbf{y}} d\mu_{\mathbf{z}} \quad . \tag{4.62}$$

This star product may be reobtained and generalized as follows. The phase Ψ given by eq. (4.52) was built without any use of the hermiticity condition. Modulo the computational assumptions introduced, its expression results essentially from the invariance conditions and one "half" of the trace condition (4.32) : the condition that leads to the factor $\delta(a_{\mathbf{y}} - a_{\mathbf{z}})$ on the right hand side of eq. (4.31). A similar computation as the one described here above, but ignoring the other "half" of the trace condition (eq. (4.54)), leads to:

$$B(a,\,b)\, B(b,\,a) = \frac{B(a,\,a)\, B(b,\,b)}{B(a-b,\,a-b)} \cosh^2[(a-b)] \qquad . \tag{4.63}$$

Note that by interchanging a and b in this equation we obtain that $B(a,\,a) = B(-a,\,-a)$. As a consequence, $B(a,a)$ is an even function and the star product satisfies in general a twisted trace condition, *i.e.* a trace condition that instead of $\delta^2[\mathbf{y} - \mathbf{z}]$ in eqs (4.28, 4.31), involves a slightly more general but nevertheless always invariant, symmetric kernel $F(n_{\mathbf{y}} - n_{\mathbf{z}})\; \delta(a_{\mathbf{y}} - a_{\mathbf{z}})$ in its right hand side, with:

$$F(n_{\mathbf{y}} - n_{\mathbf{z}}) = \frac{1}{2\,\pi \lambda\!\!\!^{-}} \int B(a,a) \mathrm{e}^{\frac{i}{\lambda\!\!\!^{-}}(n_{\mathbf{y}} - n_{\mathbf{z}}) \sinh a} da \qquad . \tag{4.64}$$

Conversely, if we fix the distribution $F(n_{\mathbf{y}} - n_{\mathbf{z}})$, $B(a,a)$ must be equal to:

$$B(a,a) = \tilde{\mathcal{F}}(a) = \cosh(a)\, \hat{\mathrm{F}}(\lambda\!\!\!^{-\,-1} \sinh(a)) \qquad , \tag{4.65}$$

where $\hat{\mathrm{F}}(k) = \int F(n) \exp(-i\, k\, n)\, dn$ is the Fourier transform of F.

The most general solution of eq. (4.63) can easily be expressed by decomposing the function $B(a,b)$ into its symmetric B_s and antisymmetric B_a parts:

$$B_s(a,b) = \frac{1}{2} \left(B(a,b) + B(b,a)\right) \qquad , \qquad B_a(a,b) = \frac{1}{2} \left(B(a,b) - B(b,a)\right) \qquad . \tag{4.66}$$

Equations (**??**)imply that the function B_a vanishes when one of its arguments is zero ($B_a(a,0) = 0$), but otherwise is arbitrary. The symmetric part of the B-function depends on the function $\tilde{\mathcal{F}}(a)$ defined in eq. (4.65), with the additional condition $\tilde{\mathcal{F}}(0) = 1$ ensuring that $\int_{-\infty}^{\infty} F(n) dn = 1$. It is given by:

$$B_s^2(a,b) = \frac{\tilde{\mathcal{F}}(a) \tilde{\mathcal{F}}(b)}{\tilde{\mathcal{F}}(a-b)} \cosh^2(a-b) + B_a^2(a,b) \tag{4.67}$$

Choosing B symmetric and real with $\tilde{\mathcal{F}}(\alpha)=1$ corresponds to the star product presented in [22], $\tilde{\mathcal{F}}(\alpha)=\cosh\alpha$ to a hermitian star product such that the trace condition is given by eqs. (4.28) and (4.30).

Let us also mention that some of the star products we have discussed here above are easily related to the Moyal-Weyl star product (denoted $*$)[4]:

$$(u*v)(\mathbf{x}) := \frac{1}{(2\pi\lambda\!\!\!^{-})^2}\int \mathrm{e}^{\frac{i}{\lambda\!\!\!^{-}}(\mathbf{x}-\mathbf{y})\wedge(\mathbf{x}-\mathbf{z})} u(\mathbf{y})\, v(\mathbf{z})\, d\mu_{\mathbf{y}} d\mu_{\mathbf{z}} \qquad , \tag{4.68}$$

via the sequence of transformations:

$$(u \overset{L}{\underset{G}{\star}} v) = T^{-1}\left[T[u]*T[v]\right] \qquad , \tag{4.69}$$

where:

$$T[u](a,\,n) := \frac{1}{2\,\pi\lambda\!\!\!^{-}}\int \mathrm{e}^{-\frac{i}{\lambda\!\!\!^{-}}\xi\, n}\mathcal{P}(\xi)\mathrm{e}^{\frac{i}{\lambda\!\!\!^{-}}\sinh(\xi)\nu} u(a,\,\nu)\, d\nu\, d\xi \qquad , \tag{4.70}$$

slightly generalizing, by an extra multiplication by the (non-vanishing) complex function $\mathcal{P}$, a similar transformation first obtained in ref. [22]. The kernel of the star product so defined is given by:

$$K^L[\alpha_{\mathbf{x}\,\mathbf{y}},\alpha_{\mathbf{x}\,\mathbf{z}};\nu_{\mathbf{y}\,\mathbf{x}},\nu_{\mathbf{z}\,\mathbf{x}}] = \frac{1}{(2\,\pi\lambda\!\!\!^{-})^2}\frac{\mathcal{P}(\alpha_{\mathbf{y}\,\mathbf{x}})\mathcal{P}(\alpha_{\mathbf{x}\,\mathbf{z}})}{\mathcal{P}(\alpha_{\mathbf{y}\,\mathbf{z}})}\cosh(\alpha_{\mathbf{y}\,\mathbf{z}})\,\exp(i\Psi) \qquad . \tag{4.71}$$

Formulas (4.69, 4.70) clarify some of the constraints imposed to the function $\mathcal{P}$: $\mathcal{P}(0)=1$ is necessary to obtain $u \overset{L}{\underset{G}{\star}} 1 = 1 \overset{L}{\underset{G}{\star}} u = u$, as otherwise we would have $u \overset{L}{\underset{G}{\star}} 1 = 1 \overset{L}{\underset{G}{\star}} u = \mathcal{P}(0)\, u$; its positivity on the real axis is necessary for the existence of T^{-1}, and it is only if $\mathcal{P}(a)=\overline{\mathcal{P}(-a)}$ that the hermiticity condition (4.57) can be fulfilled.

A tedious but elementary calculation shows that all the star products considered here may be seen as deformations of the canonical symplectic structure defined by the surface element under the $\mathcal{AN}$ group (if $\mathcal{P}(0)$ is properly normalized). Indeed, using the kernel (4.21), one finds at first order in $\lambda\!\!\!^{-}$:

$$\begin{aligned}(u \overset{L}{\underset{G}{\star}} v)(\mathbf{x}) \;&=\; u(\mathbf{x})\, v(\mathbf{x}) - i\lambda\!\!\!^{-}\, B(0,0)\, u(\mathbf{x})(\overleftarrow{\partial}_{a_{\mathbf{x}}}\overrightarrow{\partial}_{n_{\mathbf{x}}} - \overleftarrow{\partial}_{n_{\mathbf{x}}}\overrightarrow{\partial}_{a_{\mathbf{x}}})v(\mathbf{x}) \\ &+ i\lambda\!\!\!^{-}\,(v(\mathbf{x})\partial_{n_{\mathbf{x}}}u(\mathbf{x})B^{(0,1)}(0,0) - u(\mathbf{x})\partial_{n_{\mathbf{x}}}v(\mathbf{x})B^{(1,0)}(0,0)) + O(\lambda\!\!\!^{-2})\,.\end{aligned} \tag{4.72}$$

In order to define an invariant star product, the amplitude $B(a,b)$ has in particular to satisfy the relations (4.53), which force $B(0,0)=1$ and $(\partial_a B)(0,0)=(\partial_b B)(0,0)=0$. Equivalently, this implies that $\mathcal{P}(0)=1$ in the kernel (4.71).

To complete the construction of the star products, the functional space on which they are defined must be specified. Let us focus on the star products defined on $\mathcal{AN}$ which are related to the Moyal-Weyl product by an explicit intertwiner T, given by eqs. (4.69,

[4]Formally, all star products are equivalent to a Moyal-Weyl star product; the transformation T that makes the correspondence can always be constructed, step by step, as a formal series. The point here is that we have explicit transformations that allow to define the functional space on which the star product constitutes an internal composition law.

4.70). We remind that the Schwartz space $\mathcal{S}$ is stable with respect to the Moyal-Weyl star product [25]. The star product given by (4.69) will consequently define an associative algebra structure on the space $T^{-1}\mathcal{S} \subset \mathcal{S}'$, provided $T[f] \in \mathcal{S} \;\forall\; f \in \mathcal{S}$. As the Fourier transform is an isomorphism of the space $\mathcal{S}$, this will be the case if the function $\mathcal{P}(a)$ is C^∞, nowhere zero on the real axis and increases at infinity not faster than a power of $\exp(|a|)$. Unfortunately the functional space so obtained does not yet contain the constant function 1.

To overcome this difficulty, we forget, for a while, about the dependence of eq. (4.70) on the variable a and limit ourselves to the n dependence of the function. The mapping T^{-1}:

$$T^{-1}[a,\, u](\nu) := \frac{1}{2\,\pi\lambda} \int \mathrm{e}^{-\frac{i}{\lambda}\sinh(\xi)\,\nu} \mathcal{P}^{-1}(\xi) \mathrm{e}^{\frac{i}{\lambda}\xi\, n} \cosh(\xi)\; u(a,\, n)\, dn\, d\xi \qquad , \tag{4.73}$$

is well-defined as a linear injection of $\mathcal{S}_{(n)}$ (the Schwartz space of functions of the n variable) into the tempered distribution space $\mathcal{S}'$ [22], and also as an operator $T^{-1} : \mathcal{S}' \mapsto \mathcal{S}'$. Consider now the space $\mathcal{B}_{(n)}$ of smooth bounded functions with all their derivatives bounded in the variable $n \in \mathbb{R}$. This space can be seen as a subspace of $\mathcal{S}'$. Defining $\mathcal{E}_{(n)} = T^{-1}[\mathcal{B}_{(n)}]$, we obtain a deformed algebra containing the constants. Moreover, because T only affects the n-variable, $\mathcal{E}_{(n)}$ also contains the bounded functions in the a-variable. Another way of implementing the constants in our deformed algebra is to consider the unitalization $\mathbb{C} \oplus T^{-1}\mathcal{S}$.

Finally, let us notice that if we replace in the transformation (4.70) used in eq.(4.69) the sinh function by an arbitrary monotone, real, odd function Φ, running from $-\infty$ to $+\infty$, and in the B-function the cosh functions by the derivative Φ' of this function, we still obtain the kernels of hermitian star products, which are however in general no longer left invariant under $\mathcal{AN}$. In particular the quadratic associativity condition (4.55) remains satisfied as well as the trace condition if the B-function is built according to eq. (4.59). Even more general expressions of the kernel $K[\mathbf{x}, \mathbf{y}, \mathbf{z}] = \frac{1}{4\pi^2} B \exp[i\Psi]$ that provide associative star products admitting 1 as left and right unit and verifying a trace condition do exist. For instance, we have:

$$B[\mathbf{x}, \mathbf{y}, \mathbf{z}] \;=\; \sqrt{\Phi'(0)\Phi'(a_\mathbf{x} - a_\mathbf{y})\Phi'(a_\mathbf{y} - a_\mathbf{z})\Phi'(a_\mathbf{z} - a_\mathbf{x})}\phi(a_\mathbf{y})\, \phi(a_\mathbf{z}) \qquad , \tag{4.74}$$

$$\Psi[\mathbf{x}, \mathbf{y}, \mathbf{z}] \;=\; n_\mathbf{x}\Phi(a_\mathbf{y} - a_\mathbf{z})\, \phi(a_\mathbf{x}) + n_\mathbf{y}\Phi(a_\mathbf{z} - a_\mathbf{x})\, \phi(a_\mathbf{y}) + n_\mathbf{z}\Phi(a_\mathbf{x} - a_\mathbf{y})\, \phi(a_\mathbf{z}) \tag{4.75}$$

In particular, choosing the functions $\Phi(a) = \sinh(a)$ and $\phi(a) = \exp(a)$ leads to the right invariant star product.

4.3 Star products in AdS_3 and $\widetilde{\mathrm{BTZ}}$

In the previous section we have obtained star products on the $\mathcal{AN}$ group manifold. From section (4.1) we know how to construct induced star products on spaces on which this group acts, in particular on the fibers $\mathbf{S}_0$ of AdS_3 and Σ_0 in $\widetilde{\mathrm{BTZ}}$, as far as the appropriate functional spaces are specified. Of course the $\mathcal{AN}$ group also acts on AdS_3 and $\widetilde{\mathrm{BTZ}}$ [see section 2], and as consequence induces star products on them. The value of the star

product of two functions defined on these spaces, at the point of coordinates $[\rho_0, \phi_0, w_0]$, is simply given by the star products of the same functions considered as functions on the D-brane $\rho = \rho_0$, at the point of coordinates $[\phi_0, w_0]$

We now give detailed formulas for the star products induced on $\mathbf{U}$ and $\widetilde{\mathrm{BTZ}}$. For similar formulas on the D-brane $\mathbf{S}_0$ of AdS_3 and even on the D-brane Σ_0 in $\widetilde{\mathrm{BTZ}}$ [see section (3)] it is sufficient to forget about the ρ dependence here below.

The composition law on the $\mathcal{AN}$ two parameter group, is given by:

$$[a_1,\, n_1] * [a_2,\, n_2] = [a_1 + a_2, n_2 + n_1\, \exp(-a_2)] \tag{4.76}$$

from which we infer:

$$[a,\, n]^{-1} = [-a,\, -n \exp a\,] \qquad . \tag{4.77}$$

In the following, when no confusion could arise, we shall identify points of $\mathbf{U}$, $\widetilde{\mathrm{BTZ}}$ or group elements of $\mathcal{AN}$, respectively, with their triplet of coordinates or pair of parameters.

The left action of an element $(a,\, n)$ on a point is given by the mapping:

$$[\rho,\, \phi,\, w] \mapsto [\rho,\, \phi + a,\, w + n\, \exp(-\phi)] \qquad ; \tag{4.78}$$

the right action is given by:

$$[\rho,\, \phi,\, w] \mapsto [\rho,\, \phi + a,\, w\, \exp(-a) + n] \qquad . \tag{4.79}$$

Let us suppose to have a left invariant star product (4.14) on the group $\mathcal{AN}$, defined by a kernel K^L (eq. 4.17). It induces [see eq. (4.7)] on the domain $\mathbf{U}$ the star product:

$$\begin{aligned}(U \underset{\mathbf{U}}{\overset{r}{\star}} V)[\rho,\, \phi,\, w] = & \int K^L[-a_1, -a_2,\, n_1,\, n_2]\, U[\rho,\, \phi - a_1,\, w - n_1\, \exp(-\phi + a_1)] \\ & V[\rho,\, \phi - a_2,\, w - n_2\, \exp(-\phi + a_2)]\, da_1\, dn_1\, da_2\, dn_2 \qquad , \qquad (4.80) \\ = & \int K^L[-\phi + \alpha_1, -\phi + \alpha_2,\, (w - \nu_1) \exp(\alpha_1),\, (w - \nu_2) \exp(\alpha_2)]\, U[\rho,\, \alpha_1,\, \nu_1] \\ & V[\rho,\, \alpha_2,\, \nu_2]\, \exp(\alpha_1)\, d\alpha_1\, d\nu_1\, \exp(\alpha_2)\, d\alpha_2\, d\nu_2 \qquad , \qquad (4.81)\end{aligned}$$

which is invariant under the right action (4.79) on $\mathcal{AN}$ on $\mathbf{U}$. On the other hand we also obtain, from the kernel K^L, a right invariant star product on the group $\mathcal{AN}$:

$$\begin{aligned}(u \underset{\mathcal{AN}}{\overset{R}{\star}} v)[a_x,\, n_x] = & \int K^L[-a_x + a_y, -a_x + a_z,\, (n_x - n_y) \exp(a_y),\, (n_x - n_z) \exp(a_z)] \\ & u[a_y,\, n_y]\, v[a_z,\, n_z]\, \exp[a_y]\, da_y\, dn_y\, \exp[a_z]\, da_z\, dn_z \end{aligned} \tag{4.82}$$

and, as a consequence, another induced star product on $\mathbf{U}$:

$$\begin{aligned}(U \underset{\mathbf{U}}{\overset{l}{\star}} V)[\rho,\, \phi,\, w] = & \int K^L[a_1, a_2, -\, n_1\, \exp(a_1), -\, n_2\, \exp(a_2)]\, U[\rho,\, \phi - a_1,\, (\,w - n_1) \exp(a_1)] \\ & V[\rho,\, \phi - a_2,\, (w - n_2) \exp(a_2)]\, \exp(a_1)\, da_1\, dn_1\, \exp(a_2)\, da_2\, dn_2 \qquad (4.83) \\ = & \int K^L[\phi - \alpha_1, \phi - \alpha_2,\, \nu_1 - w \exp(\phi - \alpha_1),\, \nu_2 - w \exp(\phi - \alpha_2)]\, U[\rho,\, \alpha_1,\, \nu_1] \\ & V[\rho,\, \alpha_2,\, \nu_2]\, d\alpha_1\, d\nu_1\, d\alpha_2\, d\nu_2 \qquad . \qquad (4.84)\end{aligned}$$

This expression is invariant under the left action of $\mathcal{AN}$ on $\mathbf{U}$ and is obviously compatible with the quotient leading to the $\widetilde{\mathrm{BTZ}}$ space. Indeed using the global coordinate system (2.20) the identification yielding the maximally extended space from $\mathbf{U}$ reads as:

$$[\rho,\,\phi,\,w] \equiv [\rho,\,\phi+2\pi\sqrt{M},\,w] \qquad . \tag{4.85}$$

Therefore, if the functions U and V are periodic on $\mathbf{U}$, in the ϕ variable, the star product $U \underset{\mathbf{U}}{\overset{l}{\star}} V$ will also be periodic, contrary to what happens with the right invariant star product.

This allows us to define a deformed product *at the quotient level*, i.e. the star-product of two functions U and V on $\widetilde{\mathrm{BTZ}}$. To this end we adopt a method of images, that is, we extend periodically the function on $\mathbf{U}$ by defining

$$\tilde{U}[\rho,\phi,w] = \sum_{k\in\mathbb{Z}} U[\rho,\phi+k2\pi\sqrt{M},w] \tag{4.86}$$

and

$$U \underset{\widetilde{BTZ}}{\star} V \quad = \quad \tilde{U} \underset{\mathbf{U}}{\overset{l}{\star}} \tilde{V} \quad . \tag{4.87}$$

To give a precise meaning to this formal sum, one proceeds as follows. Denoting by $\pi : \mathbf{S}_0 \mapsto \Sigma_0$ the quotient map onto the $\widetilde{\mathrm{BTZ}}$ space. If C_c^∞ is the space of smooth compact supported functions on Σ_0, we notice that $\pi^\star(C_c^\infty(\Sigma_0)) \subset \mathcal{E}$ is the space of smooth ϕ periodic functions which, for fixed value of ϕ, have compact support in w. Now we observe that the mappings T and T^{-1} only involve the w variable, ϕ being a spectator variable. Knowing that the space $\mathcal{B}_{(\phi,\,w)}$ of two-variable $(\phi,\,w)$ functions is stable under the Moyal-Weyl product (4.68)[26], we find that if $U, V \in \mathcal{E}_{(\phi,\,w)} = T^{-1}[\mathcal{B}_{(\phi,\,w)}]$, $U \underset{\mathbf{U}}{\overset{l}{\star}} V$ is also in $\mathcal{E}_{(\phi,\,w)}$. The subalgebra of $\mathcal{E}_{(\phi,\,w)}$ generated by $\pi^\star(C_c^\infty(\Sigma_0))$ is then constituted by ϕ-periodic functions, owing to the $\mathcal{A}$-invariance of the star product. It is therefore identified with a function algebra on Σ_0 and on the $\widetilde{\mathrm{BTZ}}$ space.

5. Deformation of Dirac operators

Motivated by Connes' definition of noncommutative spectral triples [27], we are interested in the definition of Dirac operators associated to the deformed algebras introduced in sections 4.2 and 4.3 . Note that the left and right action of the group $\mathcal{AN}$ are crucially different. The left actions constitute an isometry group for the metric (2.20) or its restriction to the D-brane $\rho = \rho_0$. The Dirac operator on this D-brane can be written as

$$\not{D}_{\rho_0} = \mathrm{sech}\rho_0 \left[\begin{pmatrix} 0 & \partial_\phi - w\,\partial_w + \partial_w \\ \partial_\phi - w\,\partial_w - \partial_w & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \qquad . \tag{5.1}$$

Two vector fields appear naturally in this operator: $\delta_\pm = \partial_\phi - w\,\partial_w \pm \partial_w$. They constitute left invariant (isometry invariant) vector field and are the generators of the right transformations. If $\mathbf{\Psi}$ is a two component spinor field on the D-brane and u a function, the Dirac operator (5.1) behaves as a derivative in the star product algebra:

$$\not{D}_{\rho_0}\left(u \underset{\mathbf{S}_0}{\overset{r}{\star}} \mathbf{\Psi} \right) = (\not{D}_{\rho_0} u \mathbb{1}) \underset{\mathbf{S}_0}{\overset{r}{\star}} \mathbf{\Psi} + u \underset{\mathbf{S}_0}{\overset{r}{\star}} \not{D}_{\rho_0} \mathbf{\Psi} \qquad , \tag{5.2}$$

the components of the star product of a spinor field and a function being defined as the star product of the function with each of the components of the spinor field. Accordingly the commutator $[\not{D}_{\rho_0},\, u] = \not{D}_{\rho_0} u \mathbb{1}$ is a bounded operator for all Schwartz functions on $\mathbf{S}_0$. This way, we have at hand the ingredients for a noncommutative spectral triple on AdS_2 and AdS_3 spaces. Nevertheless, as discussed in the preceding section, this right invariant star product is *not* compatible with the quotient leading to the BTZ spaces.

Turning to $\widetilde{\mathrm{BTZ}}$ spaces, an immediate calculation shows that the operator (5.1) does not define a derivation on the algebra based on $\overset{l}{\underset{\mathbf{S}_0}{\star}}$. Nevertheless we can provide a deformation of the Dirac operator in an algebraic framework, analogous to those considered in ref. [28] We shall consider genuine deformations of $\not{D}$, built as follows. Derivative in the directions given by left invariant vector fields obey the Leibnitz rule with respect to the star product $\overset{r}{\underset{\mathbf{S}_0}{\star}}$ only because this star product was itself right invariant. Accordingly, right invariant vector fields will define derivatives in the algebra obtained from the left invariant star product $\overset{l}{\underset{\mathbf{S}_0}{\star}}$. The point here is that these derivative operators read as: $\mu_1 = \partial_\phi$ and $\mu_2 = \exp(-\phi)\partial_w$ and the Dirac operator expresses itself as:

$$
\begin{aligned}
\not{D}_{\rho_0} \;&=\; \mathrm{sech}\rho_0 \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_\phi + \begin{pmatrix} 0 & (1-w)\exp(\phi) \\ -(1+w)\exp(\phi) & 0 \end{pmatrix} \exp(-\phi)\partial_w \right. \\
&\qquad \left. -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \qquad (5.3)\\
&:= \Gamma^I[\phi, w]\, \mu_I + K[\phi, w] \qquad . \qquad (5.4)
\end{aligned}
$$

Actually the matrix K is constant (with respect to the ϕ and w variables), as well as Γ^1, but we may consider slightly more general situation than the simplest Dirac operator considered here, for instance by adding gauge field coupling. The idea to extend the Dirac operator as an operator (denoted $\mathbb{D}$), which on the module of functions and spinor fields with the star product as multiplication enjoys similar properties as a derivative operator, consists in defining it as follows:

$$
(\mathbb{D}\Psi)^j = \mu_I(\Psi^k) \overset{l}{\underset{\mathbf{S}_0}{\star}} \left(\Gamma^I\right)^j{}_k + \Psi^k \overset{l}{\underset{\mathbf{S}_0}{\star}} (K)^j{}_k \qquad . \qquad (5.5)
$$

We have ordered the various terms and multiply them using the star product, so that a formula analogs to eq. (5.2) remains valid :

$$
\left(\mathbb{D}_{\rho_0}(u \overset{l}{\underset{\mathbf{S}_0}{\star}} \Psi)\right)^l = u \overset{l}{\underset{\mathbf{S}_0}{\star}} (\mathbb{D}_{\rho_0}\Psi)^l + \mu_I(u) \overset{l}{\underset{\mathbf{S}_0}{\star}} \Psi^k \overset{l}{\underset{\mathbf{S}_0}{\star}} (\Gamma^I)^l_k \qquad (5.6)
$$

The motivation to push the non constant Γ matrices to the right is the requirement that the Clifford multiplication remains (left) linear. With this definition $([\mathbb{D}_{\rho_0},\, u]\Psi)^l = \mu_I(u) \overset{l}{\underset{\mathbf{S}_0}{\star}} \Psi^k \overset{l}{\underset{\mathbf{S}_0}{\star}} (\Gamma^I)^l_k$. This operator is not bounded and consequently another deformation of the Dirac operator should be used in Connes' construction. Note that, while the vector field μ_2 is not defined on $\widetilde{\mathrm{BTZ}}$ due to the appearance of the factor $\exp(\phi)$, the Dirac operator (of course) **and** its twisted version $\mathbb{D}_{\rho_0}$ are well defined.

6. Discussion

BTZ black holes exhibit interesting geometrical structures. As shown in [6], AdS_3 space admits a foliation by twisted conjugacy classes in $\mathrm{SL}(2,\mathbb{R})$, stable under the identification subgroup leading to the the massive non-rotating black hole. Each of the leaves of the foliation showed to be canonically endowed with a Poisson structure. This essentially reduces the study of the black hole to a two-dimensional problem. According to [10], these two-dimensional conjugacy classes represent D-strings in AdS_3 space-time. Consequently, they project into closed branes in the BTZ black hole background, whereas a generic D-string in AdS_3 projects onto an infinite D-string wound around the black hole. We analyzed these closed D-strings in section 3. using the (approximate) space-time description given by the DBI action.

Furthermore, the Poisson structure intrinsically defined on each leaf is the primary input for the construction of a noncommutative deformation of the usual pointwise product of functions (i.e. a star product). In view of this construction, we first showed that the non-rotating massive BTZ space-time admits two nonequivalent extensions beyond the singularities in the causal structure, much in the same way as the two-dimensional Misner space can be maximally extended. In one of these extensions, each of the leaves of the foliation can be seen as orbits of the action of the non-Abelian two parameter subgroup $\mathcal{AN}$ of $\mathrm{SL}(2,\mathbb{R})$. This action allowed us to construct a family of induced star products on the two-dimensional leaves of $\widetilde{BTZ}$ and on AdS_3. This result extends Rieffel's strict deformation theory for manifolds admitting an action of Abelian groups to manifolds with a non-Abelian group action, and was first introduced in [22] using different techniques.

This construction raises some questions about causality. Indeed, the closure of the support of the non-formal star product of two compactly supported functions generally extends beyond the closure of the intersection of the supports of the functions. However, for the star products satisfying the trace condition (4.30), we get the weaker condition that the integral of the star product of two functions always vanishes if their supports do not overlap; otherwise, the star product depends only on the values of the functions on the intersection of their supports. Of course, we recover causality if we restrict ourselves to the *formal* star products, obtained from (4.69) by extending the expansion (4.72) to all orders.

There are, in our opinion, two main directions worth being investigated. The first is related to Connes' construction (of Yang-Mills type actions). We showed in section 5 that we have at hand the first ingredients needed for a non-commutative spectral triple on AdS_2 and AdS_3 (see [29, 30] for the definition of semi-Riemannian spectral triples).

The second direction results from the close link between string theory (branes and open strings) and non-commutative geometry. In this paper, we focused on the *approximate* space-time approach of the D-branes wrapping around the BTZ black hole, using the DBI action. Actually, in the *exact* world-sheet approach, a D-brane is described in boundary conformal field theory by its interaction with closed strings, i.e. by a closed string state called boundary state. For branes in a flat background with constant B-field, it was shown in [12] (see also [31, 32]) from the operator product expansions of tachyonic open string vertex operators that the brane's world-volume geometry is given by a Moyal-Weyl

deformation of the classical algebra of functions on the brane, and scattering amplitudes of massless open string modes give rise to a non-commutative Yang-Mills theory [13, 14]. Branes in compact group manifolds reveal a more intricate structure, but non-commutative geometry also emerges due to the presence of the WZW background B-field (see for instance [33]). It could be interesting to explore the possible link between the family of star products we introduced and the worldvolume geometry of D-branes in the non-compact $SL(2,\mathbb{R})$ WZW model.

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